3.4 Concavity and the Second Derivative Test

- **Determine intervals on which a function is concave upward or concave downward.**
- **Find any points of inflection of the graph of a function.**
- **Apply the Second Derivative Test to find relative extrema of a function.**

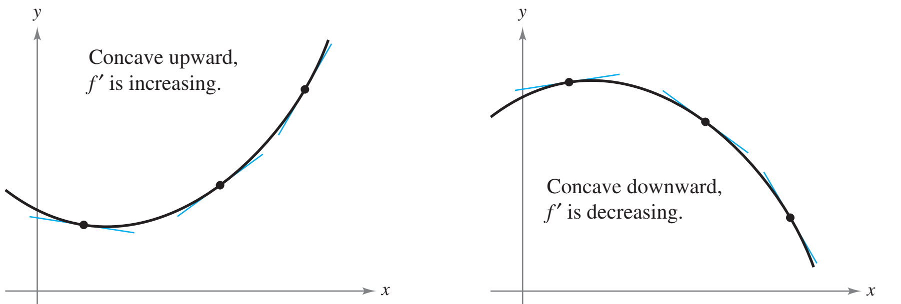
Concavity

You have already seen that locating the intervals in which a function f increases or decreases helps to describe its graph. In this section, you will see how locating the intervals in which f' increases or decreases can be used to determine where the graph of f is *curving upward* or *curving downward.*

> **Definition of Concavity**
>
> Let f be differentiable on an open interval I. The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.

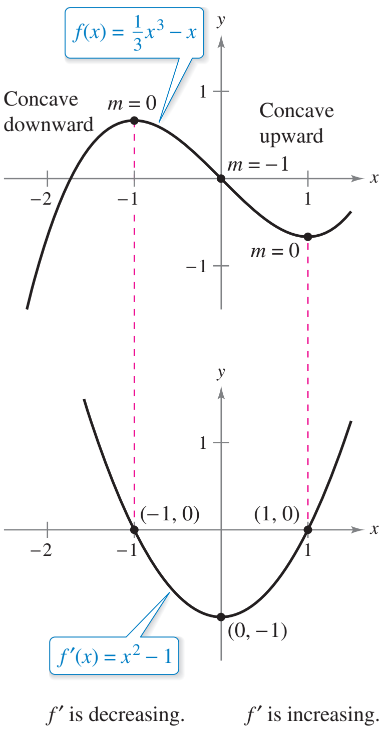
The following graphical interpretation of concavity is useful. (See Appendix A for a proof of these results.) *See LarsonCalculus.com for Bruce Edwards's video of this proof.*

1. Let f be differentiable on an open interval I. If the graph of f is concave *upward* on I, then the graph of f lies *above* all of its tangent lines on I. [See Figure 3.23(a).]
2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I. [See Figure 3.23(b).]



(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.

Figure 3.23



The concavity of f is related to the slope of the derivative.

Figure 3.24

To find the open intervals on which the graph of a function f is concave upward or concave downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$f(x) = \frac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because

$$f'(x) = x^2 - 1$$

is decreasing there. (See Figure 3.24.) Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing on $(0, \infty)$.

The next theorem shows how to use the *second* derivative of a function f to determine intervals on which the graph of f is concave upward or concave downward. A proof of this theorem follows directly from Theorem 3.5 and the definition of concavity.

REMARK A third case of Theorem 3.7 could be that if $f''(x) = 0$ for all x in I, then f is linear. Note, however, that concavity is not defined for a line. In other words, a straight line is neither concave upward nor concave downward.

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

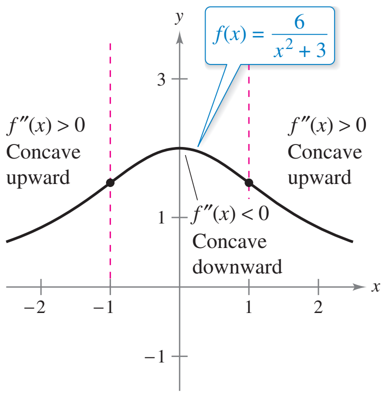
To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or f'' does not exist. Use these x-values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

EXAMPLE 1 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or downward.



From the sign of f'', you can determine the concavity of the graph of f.

Figure 3.25

Solution Begin by observing that f is continuous on the entire real number line. Next, find the second derivative of f.

$$f(x) = 6(x^2 + 3)^{-1}$$ Rewrite original function.

$$f'(x) = (-6)(x^2 + 3)^{-2}(2x)$$ Differentiate.

$$= \frac{-12x}{(x^2 + 3)^2}$$ First derivative

$$f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(x^2 + 3)(2x)}{(x^2 + 3)^4}$$ Differentiate.

$$= \frac{36(x^2 - 1)}{(x^2 + 3)^3}$$ Second derivative

Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire real number line, you should test f'' in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. The results are shown in the table and in Figure 3.25.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

The function given in Example 1 is continuous on the entire real number line. When there are x-values at which the function is not continuous, these values should be used, along with the points at which $f''(x) = 0$ or $f''(x)$ does not exist, to form the test intervals.

EXAMPLE 2 Determining Concavity

Determine the open intervals on which the graph of

$$f(x) = \frac{x^2 + 1}{x^2 - 4}$$

is concave upward or concave downward.

Solution Differentiating twice produces the following.

$$f(x) = \frac{x^2 + 1}{x^2 - 4} \qquad \text{Write original function.}$$

$$f'(x) = \frac{(x^2 - 4)(2x) - (x^2 + 1)(2x)}{(x^2 - 4)^2} \qquad \text{Differentiate.}$$

$$= \frac{-10x}{(x^2 - 4)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 - 4)^2(-10) - (-10x)(2)(x^2 - 4)(2x)}{(x^2 - 4)^4} \qquad \text{Differentiate.}$$

$$= \frac{10(3x^2 + 4)}{(x^2 - 4)^3} \qquad \text{Second derivative}$$

There are no points at which $f''(x) = 0$, but at $x = \pm 2$, the function f is not continuous. So, test for concavity in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$, as shown in the table. The graph of f is shown in Figure 3.26.

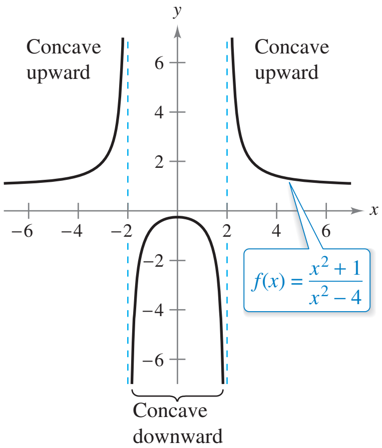


Figure 3.26

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
Test Value	$x = -3$	$x = 0$	$x = 3$
Sign of $f''(x)$	$f''(-3) > 0$	$f''(0) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of Inflection

The graph in Figure 3.25 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, then that point is a **point of inflection.** Three types of points of inflection are shown in Figure 3.27.

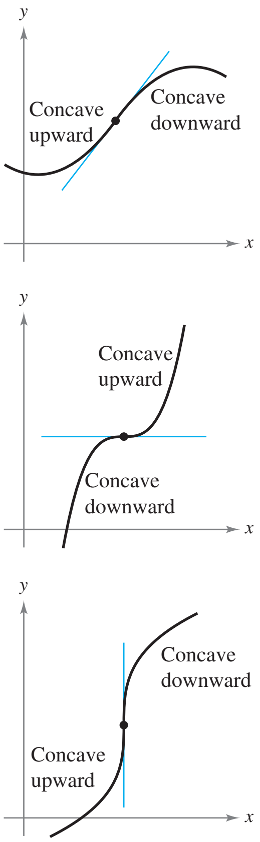
Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at the point.

REMARK The definition of *point of inflection* requires that the tangent line exists at the point of inflection. Some books do not require this. For instance, we do not consider the function

$$f(x) = \begin{cases} x^3, & x < 0 \\ x^2 + 2x, & x \geq 0 \end{cases}$$

to have a point of inflection at the origin, even though the concavity of the graph changes from concave downward to concave upward.

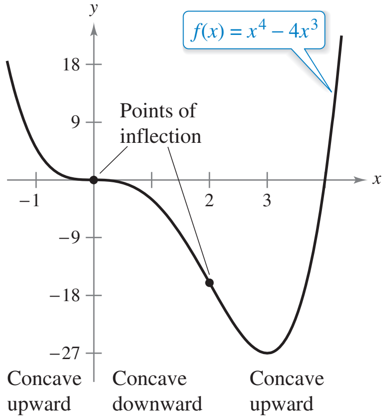


The concavity of f changes at a point of inflection. Note that the graph crosses its tangent line at a point of inflection.
Figure 3.27

To locate *possible* points of inflection, you can determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist. This is similar to the procedure for locating relative extrema of f.

THEOREM 3.8 Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$.



Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.
Figure 3.28

EXAMPLE 3 Finding Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of

$$f(x) = x^4 - 4x^3.$$

Solution Differentiating twice produces the following.

$$f(x) = x^4 - 4x^3 \qquad \text{Write original function.}$$
$$f'(x) = 4x^3 - 12x^2 \qquad \text{Find first derivative.}$$
$$f''(x) = 12x^2 - 24x = 12x(x - 2) \qquad \text{Find second derivative.}$$

Setting $f''(x) = 0$, you can determine that the possible points of inflection occur at $x = 0$ and $x = 2$. By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection. A summary of this testing is shown in the table, and the graph of f is shown in Figure 3.28.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

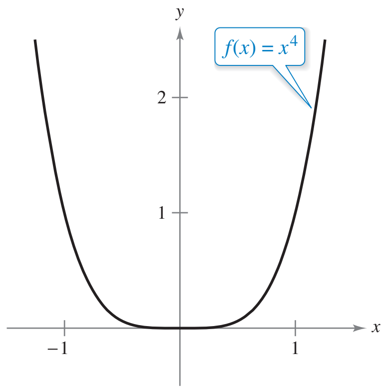
Exploration

Consider a general cubic function of the form

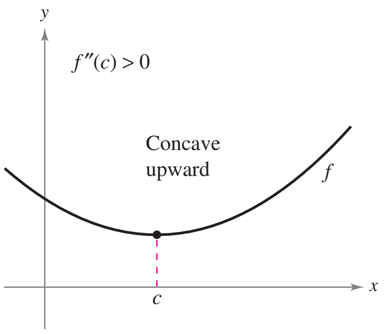
$$f(x) = ax^3 + bx^2 + cx + d.$$

You know that the value of d has a bearing on the location of the graph but has no bearing on the value of the first derivative at given values of x. Graphically, this is true because changes in the value of d shift the graph up or down but do not change its basic shape. Use a graphing utility to graph several cubics with different values of c. Then give a graphical explanation of why changes in c do not affect the values of the second derivative.

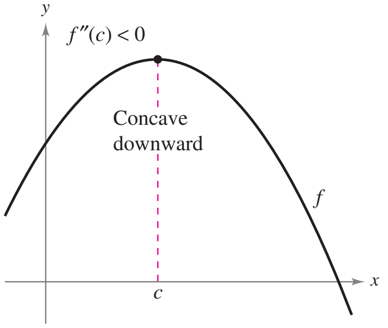
The converse of Theorem 3.8 is not generally true. That is, it is possible for the second derivative to be 0 at a point that is *not* a point of inflection. For instance, the graph of $f(x) = x^4$ is shown in Figure 3.29. The second derivative is 0 when $x = 0$, but the point $(0, 0)$ is not a point of inflection because the graph of f is concave upward in both intervals $-\infty < x < 0$ and $0 < x < \infty$.



$f''(x) = 0$, but $(0, 0)$ is not a point of inflection.
Figure 3.29



If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.



If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum.

Figure 3.30

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, then $f(c)$ must be a relative minimum of f. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, then $f(c)$ must be a relative maximum of f (see Figure 3.30).

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Proof If $f'(c) = 0$ and $f''(c) > 0$, then there exists an open interval I containing c for which

$$\frac{f'(x) - f'(c)}{x - c} = \frac{f'(x)}{x - c} > 0$$

for all $x \neq c$ in I. If $x < c$, then $x - c < 0$ and $f'(x) < 0$. Also, if $x > c$, then $x - c > 0$ and $f'(x) > 0$. So, $f'(x)$ changes from negative to positive at c, and the First Derivative Test implies that $f(c)$ is a relative minimum. A proof of the second case is left to you. *See LarsonCalculus.com for Bruce Edwards's video of this proof.*

EXAMPLE 4 Using the Second Derivative Test

See LarsonCalculus.com for an interactive version of this type of example.

Find the relative extrema of

$$f(x) = -3x^5 + 5x^3.$$

Solution Begin by finding the first derivative of f.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2)$$

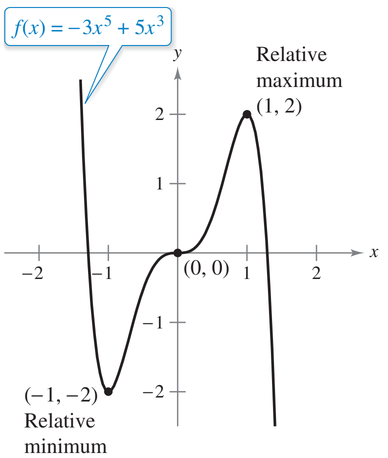
From this derivative, you can see that $x = -1, 0$, and 1 are the only critical numbers of f. By finding the second derivative

$$f''(x) = -60x^3 + 30x = 30x(1 - 2x^2)$$

you can apply the Second Derivative Test as shown below.

Point	$(-1, -2)$	$(0, 0)$	$(1, 2)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(0) = 0$	$f''(1) < 0$
Conclusion	Relative minimum	Test fails	Relative maximum

Because the Second Derivative Test fails at $(0, 0)$, you can use the First Derivative Test and observe that f increases to the left and right of $x = 0$. So, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 3.31.



(0, 0) is neither a relative minimum nor a relative maximum.

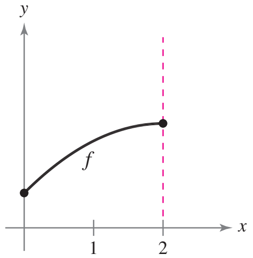
Figure 3.31

3.4 Exercises

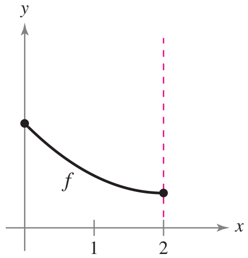
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Using a Graph In Exercises 1 and 2, the graph of f is shown. State the signs of f' and f'' on the interval $(0, 2)$.

1.



2.



Determining Concavity In Exercises 3–14, determine the open intervals on which the graph is concave upward or concave downward.

3. $y = x^2 - x - 2$

4. $g(x) = 3x^2 - x^3$

5. $f(x) = -x^3 + 6x^2 - 9x - 1$

6. $h(x) = x^5 - 5x + 2$

7. $f(x) = \dfrac{24}{x^2 + 12}$

8. $f(x) = \dfrac{2x^2}{3x^2 + 1}$

9. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

10. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$

11. $g(x) = \dfrac{x^2 + 4}{4 - x^2}$

12. $h(x) = \dfrac{x^2 - 1}{2x - 1}$

13. $y = 2x - \tan x, \quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

14. $y = x + \dfrac{2}{\sin x}, \quad (-\pi, \pi)$

Finding Points of Inflection In Exercises 15–30, find the points of inflection and discuss the concavity of the graph of the function.

15. $f(x) = x^3 - 6x^2 + 12x$

16. $f(x) = -x^3 + 6x^2 - 5$

17. $f(x) = \frac{1}{2}x^4 + 2x^3$

18. $f(x) = 4 - x - 3x^4$

19. $f(x) = x(x - 4)^3$

20. $f(x) = (x - 2)^3(x - 1)$

21. $f(x) = x\sqrt{x + 3}$

22. $f(x) = x\sqrt{9 - x}$

23. $f(x) = \dfrac{4}{x^2 + 1}$

24. $f(x) = \dfrac{x + 3}{\sqrt{x}}$

25. $f(x) = \sin \dfrac{x}{2}, \quad [0, 4\pi]$

26. $f(x) = 2 \csc \dfrac{3x}{2}, \quad (0, 2\pi)$

27. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), \quad (0, 4\pi)$

28. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

29. $f(x) = 2 \sin x + \sin 2x, \quad [0, 2\pi]$

30. $f(x) = x + 2 \cos x, \quad [0, 2\pi]$

Using the Second Derivative Test In Exercises 31–42, find all relative extrema. Use the Second Derivative Test where applicable.

31. $f(x) = 6x - x^2$

32. $f(x) = x^2 + 3x - 8$

33. $f(x) = x^3 - 3x^2 + 3$

34. $f(x) = -x^3 + 7x^2 - 15x$

35. $f(x) = x^4 - 4x^3 + 2$

36. $f(x) = -x^4 + 4x^3 + 8x^2$

37. $f(x) = x^{2/3} - 3$

38. $f(x) = \sqrt{x^2 + 1}$

39. $f(x) = x + \dfrac{4}{x}$

40. $f(x) = \dfrac{x}{x - 1}$

41. $f(x) = \cos x - x, \quad [0, 4\pi]$

42. $f(x) = 2 \sin x + \cos 2x, \quad [0, 2\pi]$

Finding Extrema and Points of Inflection Using Technology In Exercises 43–46, use a computer algebra system to analyze the function over the given interval. (a) Find the first and second derivatives of the function. (b) Find any relative extrema and points of inflection. (c) Graph f, f', and f'' on the same set of coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.

43. $f(x) = 0.2x^2(x - 3)^3, \quad [-1, 4]$

44. $f(x) = x^2\sqrt{6 - x^2}, \quad [-\sqrt{6}, \sqrt{6}]$

45. $f(x) = \sin x - \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x, \quad [0, \pi]$

46. $f(x) = \sqrt{2x} \sin x, \quad [0, 2\pi]$

WRITING ABOUT CONCEPTS

47. **Sketching a Graph** Consider a function f such that f' is increasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

48. **Sketching a Graph** Consider a function f such that f' is decreasing. Sketch graphs of f for (a) $f' < 0$ and (b) $f' > 0$.

49. **Sketching a Graph** Sketch the graph of a function f that does *not* have a point of inflection at $(c, f(c))$ even though $f''(c) = 0$.

50. **Think About It** S represents weekly sales of a product. What can be said of S' and S'' for each of the following statements?

(a) The rate of change of sales is increasing.

(b) Sales are increasing at a slower rate.

(c) The rate of change of sales is constant.

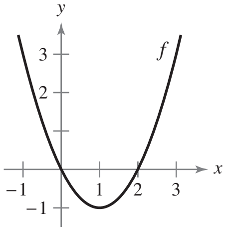
(d) Sales are steady.

(e) Sales are declining, but at a slower rate.

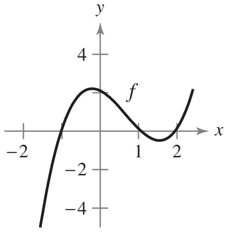
(f) Sales have bottomed out and have started to rise.

Sketching Graphs In Exercises 51 and 52, the graph of f is shown. Graph f, f', and f'' on the same set of coordinate axes. To print an enlarged copy of the graph, go to *MathGraphs.com*.

51.



52.



Think About It **In Exercises 53–56, sketch the graph of a function f having the given characteristics.**

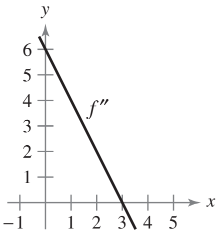
53. $f(2) = f(4) = 0$
$f'(x) < 0$ for $x < 3$
$f'(3)$ does not exist.
$f'(x) > 0$ for $x > 3$
$f''(x) < 0, x \neq 3$

54. $f(0) = f(2) = 0$
$f'(x) > 0$ for $x < 1$
$f'(1) = 0$
$f'(x) < 0$ for $x > 1$
$f''(x) < 0$

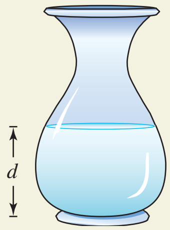
55. $f(2) = f(4) = 0$
$f'(x) > 0$ for $x < 3$
$f'(3)$ does not exist.
$f'(x) < 0$ for $x > 3$
$f''(x) > 0, x \neq 3$

56. $f(0) = f(2) = 0$
$f'(x) < 0$ for $x < 1$
$f'(1) = 0$
$f'(x) > 0$ for $x > 1$
$f''(x) > 0$

57. Think About It The figure shows the graph of f''. Sketch a graph of f. (The answer is not unique.) To print an enlarged copy of the graph, go to *MathGraphs.com*.



58. HOW DO YOU SEE IT? Water is running into the vase shown in the figure at a constant rate.



(a) Graph the depth d of water in the vase as a function of time.
(b) Does the function have any extrema? Explain.
(c) Interpret the inflection points of the graph of d.

59. Conjecture Consider the function

$$f(x) = (x - 2)^n.$$

(a) Use a graphing utility to graph f for $n = 1, 2, 3$, and 4. Use the graphs to make a conjecture about the relationship between n and any inflection points of the graph of f.
(b) Verify your conjecture in part (a).

60. Inflection Point Consider the function $f(x) = \sqrt[3]{x}$.
(a) Graph the function and identify the inflection point.
(b) Does $f''(x)$ exist at the inflection point? Explain.

Finding a Cubic Function **In Exercises 61 and 62, find a, b, c, and d such that the cubic**

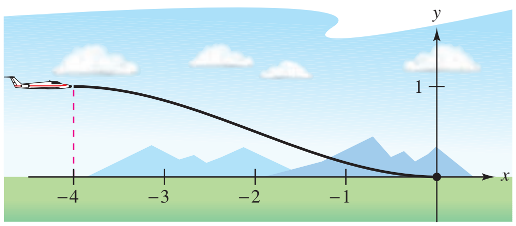
$$f(x) = ax^3 + bx^2 + cx + d$$

satisfies the given conditions.

61. Relative maximum: (3, 3)
Relative minimum: (5, 1)
Inflection point: (4, 2)

62. Relative maximum: (2, 4)
Relative minimum: (4, 2)
Inflection point: (3, 3)

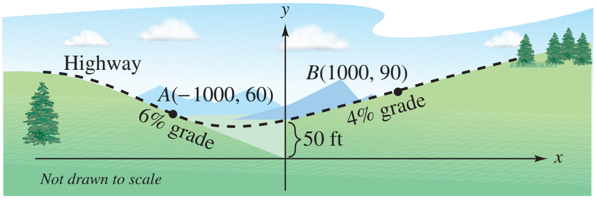
63. Aircraft Glide Path A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).



(a) Find the cubic $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$ that describes a smooth glide path for the landing.
(b) The function in part (a) models the glide path of the plane. When would the plane be descending at the greatest rate?

FOR FURTHER INFORMATION For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in *The American Mathematical Monthly*. To view this article, go to *MathArticles.com*.

64. Highway Design A section of highway connecting two hillsides with grades of 6% and 4% is to be built between two points that are separated by a horizontal distance of 2000 feet (see figure). At the point where the two hillsides come together, there is a 50-foot difference in elevation.



(a) Design a section of highway connecting the hillsides modeled by the function

$$f(x) = ax^3 + bx^2 + cx + d, \quad -1000 \le x \le 1000.$$

At points A and B, the slope of the model must match the grade of the hillside.
(b) Use a graphing utility to graph the model.
(c) Use a graphing utility to graph the derivative of the model.
(d) Determine the grade at the steepest part of the transitional section of the highway.

65. Average Cost A manufacturer has determined that the total cost C of operating a factory is

$$C = 0.5x^2 + 15x + 5000$$

where x is the number of units produced. At what level of production will the average cost per unit be minimized? (The average cost per unit is C/x.)

66. Specific Gravity A model for the specific gravity of water S is

$$S = \frac{5.755}{10^8}T^3 - \frac{8.521}{10^6}T^2 + \frac{6.540}{10^5}T + 0.99987,\ 0 < T < 25$$

where T is the water temperature in degrees Celsius.

(a) Use the second derivative to determine the concavity of S.

(b) Use a computer algebra system to find the coordinates of the maximum value of the function.

(c) Use a graphing utility to graph the function over the specified domain. (Use a setting in which $0.996 \le S \le 1.001$.)

(d) Estimate the specific gravity of water when $T = 20°$.

67. Sales Growth The annual sales S of a new product are given by

$$S = \frac{5000t^2}{8 + t^2}, \quad 0 \le t \le 3$$

where t is time in years.

(a) Complete the table. Then use it to estimate when the annual sales are increasing at the greatest rate.

t	0.5	1	1.5	2	2.5	3
S						

(b) Use a graphing utility to graph the function S. Then use the graph to estimate when the annual sales are increasing at the greatest rate.

(c) Find the exact time when the annual sales are increasing at the greatest rate.

68. Modeling Data The average typing speed S (in words per minute) of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is

$$S = \frac{100t^2}{65 + t^2}, \quad t > 0.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Use the second derivative to determine the concavity of S. Compare the result with the graph in part (a).

(c) What is the sign of the first derivative for $t > 0$? By combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases?

Linear and Quadratic Approximations **In Exercises 69–72, use a graphing utility to graph the function. Then graph the linear and quadratic approximations**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = a$. How do the approximations change as you move farther away from $x = a$?

	Function	Value of a
69.	$f(x) = 2(\sin x + \cos x)$	$a = \frac{\pi}{4}$
70.	$f(x) = 2(\sin x + \cos x)$	$a = 0$
71.	$f(x) = \sqrt{1 - x}$	$a = 0$
72.	$f(x) = \frac{\sqrt{x}}{x - 1}$	$a = 2$

73. Determining Concavity Use a graphing utility to graph

$$y = x \sin\frac{1}{x}.$$

Show that the graph is concave downward to the right of

$$x = \frac{1}{\pi}.$$

74. Point of Inflection and Extrema Show that the point of inflection of

$$f(x) = x(x - 6)^2$$

lies midway between the relative extrema of f.

True or False? **In Exercises 75–78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. The graph of every cubic polynomial has precisely one point of inflection.

76. The graph of

$$f(x) = \frac{1}{x}$$

is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$.

77. If $f'(c) > 0$, then f is concave upward at $x = c$.

78. If $f''(2) = 0$, then the graph of f must have a point of inflection at $x = 2$.

Proof **In Exercises 79 and 80, let f and g represent differentiable functions such that $f'' \neq 0$ and $g'' \neq 0$.**

79. Show that if f and g are concave upward on the interval (a, b), then $f + g$ is also concave upward on (a, b).

80. Prove that if f and g are positive, increasing, and concave upward on the interval (a, b), then fg is also concave upward on (a, b).